# Data-driven Modeling and Optimization of Dissipative Dynamics

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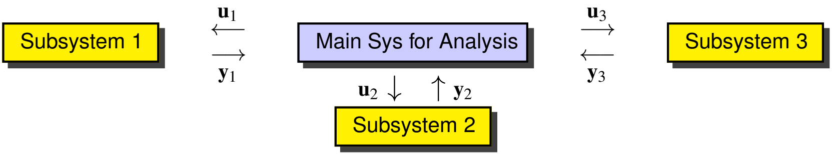
# Goals of Model Reduction




- Replace high-order complex subsystems with low-order, (but high-fidelity) surrogates. Encode high resolution/fine grain structure of the subsystem response acquired offline into compact, efficient online surrogates.
- Avoid using (expensive) human resources. Want the process to be (relatively) automatic and capable of producing reliable high-fidelity surrogates.
- Should respect underlying “physics” High fidelity may not be enough - surrogate models should behave “physically” and respect underlying conservation laws.

# Energy-based modeling of dynamical systems

**DynSys:** $\mathbf{u}(t) \in \mathbb{U} \longrightarrow \boxed{\begin{array}{ll} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}(t) & \\ & \mathbf{x}(t) \in \mathbb{X} \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x} & \end{array}} \longrightarrow \mathbf{y}(t) \in \mathbb{Y}$

- Assume: linear, time-invariant, asymp stable, min sys realization.

*Energy-based modeling*: allows for the system to extract, store, and return value ("energy") to/from the environment.
(inspired by: "Gibbs free energy", "available work", "karma" ...)

**Key Modeling Element:**

- *Supply Rate*, $w : \mathbb{Y} \times \mathbb{U} \to \mathbb{R}$ with $w(\mathbf{y}(\cdot), \mathbf{u}(\cdot)) \in \mathcal{L}^1_{loc}$
  $w(\mathbf{y}(t), \mathbf{u}(t))$ models the instantaneous exchange of value/energy of the system with the environment via inputs and outputs.

# Supply rates and dissipativity

Examples of supply rates:

- $w(\mathbf{y}(t), \mathbf{u}(t)) = \mathbf{u}(t)^T \mathbf{y}(t)$ (work ⇒ "Passive systems")
- $w(\mathbf{y}(t), \mathbf{u}(t)) = \frac{1}{2}(\|\mathbf{u}(t)\|^2 - \|\mathbf{y}(t)\|^2)$ $\left( \begin{array}{l} \text{instantaneous gain} \Rightarrow \\ \text{"Contractive systems"} \end{array} \right)$
- $w(\mathbf{y}(t), \mathbf{u}(t)) = \frac{1}{2}\left( \mathbf{y}(t) \quad \mathbf{u}(t) \right) \begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix} \begin{pmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{pmatrix}$

  with $\mathbf{M} \geq 0 \quad \mathbf{N} \geq 0$ (General quadratic supply rate)

For a given energy/value supply rate, $w(\mathbf{y}(\cdot), \mathbf{u}(\cdot))$,
a dynamical system is ***dissipative with respect to*** $w$, if
whenever the system starts in an equilibrium state at $t_0 = 0$,

$$\int_0^t w(\mathbf{y}(t), \mathbf{u}(t))\, dt \geq 0 \qquad \text{for all } t \geq 0$$

Starting from equilibrium, a dissipative system can never lose more energy to the environment than it has gained.

# Dissipative systems can store energy (but maybe not give it back)

A *storage function* associated with the supply rate, $w$, is a scalar-valued function of state, $\mathsf{H} : \mathbb{X} \to \mathbb{R}^+$, that satisfies for any $0 \leq t_0 < t_1$

$$\mathsf{H}(\mathbf{x}(t_1)) - \mathsf{H}(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} w(\mathbf{y}(t), \mathbf{u}(t))\, dt \quad \text{(dissipation inequality)}$$

- $\mathsf{H}(\mathbf{x})$ is a measure of "internal energy" in the system when it is in state $\mathbf{x}$.
- The dissipation inequality asserts the change in internal energy cannot exceed the net energy absorbed or delivered by the system from/to the environment.
- Dissipative systems cannot create "energy" internally apart from what is delivered from the environment.

- Dissipativity is an exogenous system property externally characterized; **dependent** on supply rate but **independent** of system realization.
- Storage functions are endogenous to a system internally characterized; **dependent** both on supply rate and system realization.
- For dissipative systems with linear dynamics, supply rates that are quadratic wrt input/output imply (wlog) quadratic storage functions.

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \tfrac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix} \begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} \text{ with } \mathbf{M} \geq 0, \mathbf{N} \geq 0$$

$$\Longrightarrow \qquad \mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \quad \text{for some } \mathbf{Q} > 0$$

# State-space conditions for dissipativity

Take the supply rate to be a general quadratic:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \tfrac{1}{2}\begin{pmatrix}\mathbf{y}^T & \mathbf{u}^T\end{pmatrix}\begin{bmatrix}-\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M}\end{bmatrix}\begin{pmatrix}\mathbf{y}\\ \mathbf{u}\end{pmatrix} \text{ with } \mathbf{M}\geq 0, \mathbf{N}\geq 0$$

and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x} \;\text{ for }\; \mathbf{Q} > 0.$$

- The dissipation inequality implies

$$\frac{d}{dt}\mathsf{H}(\mathbf{x}(t)) \leq w(\mathbf{y}(t),\mathbf{u}(t)).$$

$$\Longrightarrow\; \mathbf{x}^T\mathbf{Q}\dot{\mathbf{x}} = \mathbf{x}^T\mathbf{Q}(\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}) \leq \tfrac{1}{2}(\mathbf{x}^T\mathbf{C}^T\;\mathbf{u}^T)\begin{bmatrix}-\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M}\end{bmatrix}\begin{pmatrix}\mathbf{C}\mathbf{x}\\ \mathbf{u}\end{pmatrix}$$

$$\Longrightarrow\; \tfrac{1}{2}\mathbf{x}^T(\mathbf{Q}\mathbf{A}+\mathbf{A}^T\mathbf{Q}+\mathbf{C}^T\mathbf{N}\mathbf{C})\mathbf{x} + \mathbf{x}^T(\mathbf{Q}\mathbf{B}-\mathbf{C}^T\Omega)\mathbf{u} - \tfrac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u} \leq 0$$

- The system is dissipative wrt the supply $w$ if and only if the LMI

$$\begin{bmatrix}\mathbf{Q}\,\mathbf{A}+\mathbf{A}^T\,\mathbf{Q}+\mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B}-\mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q}-\Omega^T\mathbf{C} & -\mathbf{M}\end{bmatrix} \leq 0$$

has a positive-definite solution matrix, $\mathbf{Q} > 0$.

# Special case: Passive systems

Take the supply rate to be:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \mathbf{u}(t)^T\mathbf{y}(t) = \tfrac{1}{2}\left(\ \mathbf{y}^T\ \mathbf{u}^T\ \right)\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$$

(defining $\mathbf{M} = \mathbf{N} = 0$ and $\Omega = \mathbf{I}$)

and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x} \ \text{ for } \ \mathbf{Q} > 0.$$

- The system is **passive** with the storage function $\mathsf{H}(\mathbf{x})$, if and only if $\mathbf{Q}$ is a positive-definite solution to the LMI:

$$\begin{bmatrix} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\mathbf{Q} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T \\ \mathbf{B}^T\mathbf{Q} - \mathbf{C} & 0 \end{bmatrix} \leq 0 \Leftrightarrow \begin{array}{c} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\mathbf{Q} \leq 0 \\ \mathbf{Q}\,\mathbf{B} = \mathbf{C}^T \end{array} \quad \text{(Luré LMI)}$$

- Passive systems have **port-Hamiltonian** realizations. Take $\mathbf{Q}\,\mathbf{A} = \mathbf{J} - \mathbf{R}$ with $\mathbf{J} = -\mathbf{J}^T$ and $\mathbf{R} = \mathbf{R}^T$ (skew-symm + symm).

$$\begin{array}{l} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \Leftrightarrow \mathbf{Q}\dot{\mathbf{x}} = \mathbf{Q}\mathbf{A}\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u} \\ \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\mathbf{Q} = -2\mathbf{R} \leq 0 \Leftrightarrow \ \mathbf{R} \geq 0 \end{array} \Leftrightarrow \begin{array}{c} \mathbf{Q}\,\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\mathbf{x} + \mathbf{C}^T\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{array}$$

# The Plan

- Dissipative systems have realizations that encode energy flux constraints determined by the supply rate and the underlying dissipation framework via linear matrix inequalities (LMIs).
- We seek model reduction strategies that preserve this structure $\Longrightarrow$ Create reduced order surrogate models that have high fidelity and respect original dissipation constraints (this is sensible because dissipation is an exogenous property).
- BUT direct use of LMIs can be computationally untenable due to high model order and inaccessibility of internal dynamics.

Find high fidelity reduced order models that are dissipative while matching observations of true system response.

- Take advantage of interpolatory model reduction strategies:
  - Data driven reduction methods producing $\mathcal{H}_2$-optimal models.
  - Deploy convex optimization methods on low order model classes (low order LMIs constrained by observations)

# Preserving dissipativity in reduced order models

Take the supply rate to be a general quadratic:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \tfrac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix}\begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} \text{ with } \mathbf{M} \geq 0, \mathbf{N} \geq 0$$

and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ for } \mathbf{Q} > 0.$$

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}} \longrightarrow \boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}}$$

(Original Realization) (Dissipative Realization)

- $\mathbf{Q}\,\mathbf{A} = \mathbf{J} - \mathbf{R}$ with $\mathbf{J} = -\mathbf{J}^T$ and $\mathbf{R} = \mathbf{R}^T$ (skew-symm + symm).
- "Project dynamics" by approximating $\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$:

$$\mathbf{V}_r^T\mathbf{Q}\left(\mathbf{V}_r\dot{\mathbf{x}}_r(t) - \mathbf{A}\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{B}\mathbf{u}(t)\right) = 0 \quad \text{(Petrov-Galerkin)}$$

or equivalently,

$$\mathbf{V}_r^T\left(\mathbf{Q}\mathbf{V}_r\dot{\mathbf{x}}_r(t) - (\mathbf{J}-\mathbf{R})\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{Q}\mathbf{B}\mathbf{u}(t)\right) = 0 \quad \text{(Ritz-Galerkin)}$$

for some choice of subspace $\mathcal{V}_r = \mathrm{Ran}(\mathbf{V}_r)$.

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and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ for } \mathbf{Q} > 0.$$

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}\end{aligned}} \longrightarrow \boxed{\begin{aligned}\mathbf{Q}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}\end{aligned}}$$

(Original Realization) (Dissipative Realization)

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for some choice of subspace $\mathcal{V}_r = \mathsf{Ran}(\mathbf{V}_r)$.

# Preserving dissipativity in reduced order models

$$\boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}} \longrightarrow \boxed{\begin{aligned} \mathbf{Q}_r\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\mathbf{x}_r + \mathbf{Q}_r\mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r \end{aligned}}$$

(Dissipative realization) (Reduced dissipative model)

- $\mathbf{V}_r^T\left(\mathbf{Q}\mathbf{V}_r\dot{\mathbf{x}}_r(t) - (\mathbf{J}-\mathbf{R})\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{Q}\mathbf{B}\mathbf{u}(t)\right) = 0$ (Ritz-Galerkin) for some choice of subspace $\mathcal{V}_r = \text{Ran}(\mathbf{V}_r)$.
- Leads to a reduced model defined by
$$\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \quad \mathbf{J}_r = \mathbf{V}_r^T\mathbf{J}\mathbf{V}_r, \quad \mathbf{R}_r = \mathbf{V}_r^T\mathbf{R}\mathbf{V}_r,$$
$$\mathbf{C}_r = \mathbf{C}\mathbf{V}_r, \quad \mathbf{B}_r = \mathbf{Q}_r^{-1}\mathbf{V}_r^T\mathbf{Q}\mathbf{B}$$

Is this reduced model dissipative with respect to the same supply rate ?

The reduced model is defined by

$$\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \quad \mathbf{J}_r = \mathbf{V}_r^T\mathbf{J}\mathbf{V}_r, \quad \mathbf{R}_r = \mathbf{V}_r^T\mathbf{R}\mathbf{V}_r, \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}_r, \quad \mathbf{B}_r = \mathbf{Q}_r^{-1}\mathbf{V}_r^T\mathbf{Q}\mathbf{B}$$

- Evidently, $\mathbf{Q}_r > 0$, $\mathbf{J}_r = -\mathbf{J}_r^T$ and $\mathbf{R}_r = \mathbf{R}_r^T$.

The original storage, $\mathbf{Q} > 0$, solves

$$0 \geq \begin{bmatrix} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\,\mathbf{Q} + \mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q} - \Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix} = \begin{bmatrix} -2\mathbf{R} + \mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q} - \Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix}$$

$$\Longrightarrow \qquad 0 \geq \begin{bmatrix} \mathbf{V}_r^T & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -2\mathbf{R} + \mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q} - \Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{V}_r & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

$$= \begin{bmatrix} -2\mathbf{R}_r + \mathbf{C}_r^T\mathbf{N}\mathbf{C}_r & \mathbf{Q}_r\,\mathbf{B}_r - \mathbf{C}_r^T\Omega \\ \mathbf{B}_r^T\,\mathbf{Q}_r - \Omega^T\mathbf{C}_r & -\mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_r\,\mathbf{A}_r + \mathbf{A}_r{}^T\,\mathbf{Q}_r + \mathbf{C}_r^T\mathbf{N}\mathbf{C}_r & \mathbf{Q}_r\,\mathbf{B}_r - \mathbf{C}_r^T\Omega \\ \mathbf{B}_r^T\,\mathbf{Q}_r - \Omega^T\mathbf{C}_r & -\mathbf{M} \end{bmatrix}$$

- Thus, $\mathbf{R}_r \geq 0$ and $\mathbf{A}_r = \mathbf{Q}_r^{-1}(\mathbf{J}_r - \mathbf{R}_r)$ is asymp stable.
- $\Rightarrow$ The reduced system will be dissipative for *any* choice of $\mathcal{V}_r$

# Finding effective reduced order dissipative models

$$\boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}} \longrightarrow \boxed{\begin{aligned} \mathbf{Q}_r\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\mathbf{x}_r + \widehat{\mathbf{B}}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r \end{aligned}}$$

(Original dissipative realization) (Reduced dissipative realization)

- Fourier Transforms: $\mathbf{u}(t) \mapsto \hat{\mathbf{u}}(\omega)$, $\quad \mathbf{y}(t) \mapsto \hat{\mathbf{y}}(\omega)$

  Original response: $\hat{\mathbf{y}}(\omega) = \mathcal{G}(\imath\omega)\hat{\mathbf{u}}(\omega)$

  Reduced response: $\hat{\mathbf{y}}_r(\omega) = \mathcal{G}_r(\imath\omega)\hat{\mathbf{u}}(\omega)$

  with transfer functions:

  $\mathcal{G}(s) = \mathbf{C}(s\mathbf{Q} - (\mathbf{J}-\mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}$ and $\mathcal{G}_r(s) = \mathbf{C}_r(s\mathbf{Q}_r - (\mathbf{J}_r-\mathbf{R}_r))^{-1}\widehat{\mathbf{B}}_r$.

- $$\hat{\mathbf{y}}(\omega) - \hat{\mathbf{y}}_r(\omega) = \Big(\mathcal{G}(\imath\omega) - \mathcal{G}_r(\imath\omega)\Big)\hat{\mathbf{u}}(\omega)$$

Find a modeling space $\mathcal{V}_r$ so that $\mathcal{G}_r(\imath\omega) \approx \mathcal{G}(\imath\omega)$ for $\omega \in \mathbb{R}$.

Construct a modeling subspace $\mathcal{V}_r$ that forces interpolation.

## Interpolatory projections that preserve dissipativity

Given interpolation points $\sigma_1, ..., \sigma_r$ and

tangent directions $\mathbb{b}_1, ..., \mathbb{b}_r$, construct

$$\mathbf{V}_r = [(\sigma_1 \mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_1, \ldots, (\sigma_r \mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_r].$$

Then with $\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \quad \mathbf{J}_r = \mathbf{V}_r^T\mathbf{J}\mathbf{V}_r, \quad \mathbf{R}_r = \mathbf{V}_r^T\mathbf{R}\mathbf{V}_r, \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}_r, \quad \mathbf{Q}_r\mathbf{B}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{B}$

the reduced model, $\mathcal{G}_r : \begin{cases} \mathbf{Q}_r\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\mathbf{x}_r + \mathbf{Q}_r\mathbf{B}_r\,\mathbf{u}, \\ \mathbf{y}_r = \mathbf{C}_r\mathbf{x}_r \end{cases}$

is stable, minimal, dissipative wrt the given supply rate, $w$,

and $\quad \mathcal{G}_r(\sigma_i)\mathbb{b}_i = \mathcal{G}(\sigma_i)\mathbb{b}_i \quad$ for $\; i = 1, ..., r$ .

# Data-driven interpolatory dissip-preserving MOR

What we have so far:

- Dissipativity-preserving model reduction method built on interpolatory projections with potential for high-fidelity (GOOD)
- Method is intrusive (requires explicit access to a standard realization and "internal dynamics"); involves explicit construction of a modeling subspace ($\mathbf{V}_r$). (BAD)
- Method requires knowledge of a storage ($\mathbf{Q}$) compatible with the supply rate; involves solution of a large-scale LMI (VERY BAD) .

We want a noninvasive "data-driven" approach that depends only on observed system response - idealized as transfer function evaluations. (This is consistent with data from some types of empirical testing rigs.)

## Implicit construction of interpolants

Given interp points $\{\sigma_1, ..., \sigma_r\} \subset \mathbb{C}^+$ and tang direct $\mathbb{b}_1, ..., \mathbb{b}_r$, recall

$$\mathbf{V}_r = [(\sigma_1 \mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_1, \ldots, (\sigma_r \mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_r].$$

Define $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ and $\mathbb{B}_r = [\mathbb{b}_1, \ldots, \mathbb{b}_r]$.

- $\mathbf{C}_r = \mathbf{C}\mathbf{V}_r = [\mathcal{G}(\sigma_1)\mathbb{b}_1, ...\mathcal{G}(\sigma_r)\mathbb{b}_r] \overset{\text{def}}{=} \mathbf{G}_r$
- $\mathbf{V}_r$ is the unique solution to the Sylvester equation:
  $$\mathbf{Q}\mathbf{V}_r\Sigma - (\mathbf{J} - \mathbf{R})\mathbf{V}_r = \mathbf{Q}\mathbf{B}\mathbb{B}_r$$
- Premultiply by $\mathbf{V}_r^T$:
  $$\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r\Sigma - \mathbf{V}_r^T(\mathbf{J} - \mathbf{R})\mathbf{V}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{B}\mathbb{B}_r = \widehat{\mathbf{B}}_r\mathbb{B}_r$$

  where $\widehat{\mathbf{B}}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{B} = \mathbf{Q}_r\mathbf{B}_r$

$$\boxed{\mathbf{Q}_r\Sigma - (\mathbf{J}_r - \mathbf{R}_r) = \widehat{\mathbf{B}}_r\mathbb{B}_r}$$

Interpolation Conditions

$$\boxed{\mathbf{C}_r = [\mathcal{G}(\sigma_1)\mathbb{b}_1, ...\mathcal{G}(\sigma_r)\mathbb{b}_r] \overset{\text{def}}{=} \mathbf{G}_r}$$

Data

(From before: $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ and $\mathbb{B}_r = [\mathbb{b}_1, \ldots, \mathbb{b}_r]$.)

$$\boxed{\mathbf{Q}_r\Sigma - (\mathbf{J}_r - \mathbf{R}_r) = \widehat{\mathbf{B}}_r\mathbb{B}_r}$$

Interpolation Conditions

$$\boxed{\mathbf{C}_r = [\mathcal{G}(\sigma_1)\mathbb{b}_1, \ldots \mathcal{G}(\sigma_r)\mathbb{b}_r] \stackrel{\text{def}}{=} \mathbf{G}_r}$$

Data

- **Any** choice of matrices $\mathbf{Q}_r$, $\mathbf{J}_r$, $\mathbf{R}_r$, and $\widehat{\mathbf{B}}_r$ that satisfy $\mathbf{Q}_r\Sigma - (\mathbf{J}_r - \mathbf{R}_r) = \widehat{\mathbf{B}}_r\mathbb{B}_r$ will define a reduced model

$$\mathcal{G}_r(s) = \mathbf{G}_r(s\,\mathbf{Q}_r - (\mathbf{J}_r - \mathbf{R}_r))^{-1}\widehat{\mathbf{B}}_r$$

that will interpolate the data:

$$\mathcal{G}(\sigma_i)\mathbb{b}_i = \mathcal{G}_r(\sigma_i)\mathbb{b}_i \quad \text{for } i = 1, \ldots, r$$

- Can eliminate $\mathbf{J}_r$ by taking Hermitian part.

Equivalent conditions: $\boxed{\bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + 2\mathbf{R}_r = \widehat{\mathbf{B}}_r\mathbb{B}_r + \mathbb{B}_r^*\widehat{\mathbf{B}}_r^*}$

Impose dissipativity constraints

# Data-driven interpolation for dissipative systems

Given $\Sigma$, $\mathbb{B}_r$, and $\mathbf{G}_r$ (data), find $\mathbf{Q}_r$, $\mathbf{R}_r$, and $\widehat{\mathbf{B}}_r$ that satisfy:

$$\boxed{\mathbf{Q}_r > 0}$$

$$\boxed{\bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + 2\mathbf{R}_r = \widehat{\mathbf{B}}_r\mathbb{B}_r + \mathbb{B}_r^*\widehat{\mathbf{B}}_r^*}$$

$$\boxed{\begin{bmatrix} -2\mathbf{R}_r + \mathbf{G}_r^T\mathbf{N}\mathbf{G}_r & \widehat{\mathbf{B}}_r - \mathbf{G}_r^T\Omega \\ \widehat{\mathbf{B}}_r^T - \Omega^T\mathbf{G}_r & -\mathbf{M} \end{bmatrix} \leq 0}$$

- $\mathcal{G}_r(s) = \mathbf{G}_r(s\,\mathbf{Q}_r - (\mathbf{J}_r - \mathbf{R}_r))^{-1}\widehat{\mathbf{B}}_r$ with $\mathbf{J}_r = \mathbf{Q}_r\Sigma + \mathbf{R}_r - \widehat{\mathbf{B}}_r\mathbb{B}_r$ is a stable, minimal model that is dissipative wrt the given supply rate and $\mathcal{G}_r(\sigma_i)\mathbb{b}_i = \mathcal{G}(\sigma_i)\mathbb{b}_i$ for $i = 1, ..., r$.
- Only an $\mathcal{O}(r)$ LMI is involved (cheap !) .
- This provides a computable *necessary* condition for the data to been produced by a system that is dissipative with respect to the given supply rate.

# Special case: passive (pH) systems

The supply rate associated with passivity is:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \mathbf{u}(t)^T\mathbf{y}(t) = \tfrac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$$

Given interp points $\{\sigma_1,...,\sigma_r\} \subset \mathbb{C}^+$ and tang direct $\mathbb{b}_1,...,\mathbb{b}_r$, define $\Sigma = \text{diag}(\sigma_1,...,\sigma_r)$, $\mathbb{B}_r = [\mathbb{b}_1,\ldots,\mathbb{b}_r]$, and $\mathbf{G}_r = [\mathcal{G}(\sigma_1)\mathbb{b}_1,...\mathcal{G}(\sigma_r)\mathbb{b}_r]$.

A passive interpolatory model requires:

$$\mathbf{Q}_r > 0 \quad \text{and} \quad \begin{cases} \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + 2\mathbf{R}_r = \widehat{\mathbf{B}}_r\mathbb{B}_r + \mathbb{B}_r^*\widehat{\mathbf{B}}_r^* \\ \begin{bmatrix} -2\mathbf{R}_r & \widehat{\mathbf{B}}_r - \mathbf{G}_r^* \\ \widehat{\mathbf{B}}_r^* - \mathbf{G}_r & \mathbf{0} \end{bmatrix} \leq 0 \end{cases}$$

Equivalently,

- The dissipitivity LMI reduces to: $\mathbf{R}_r \geq 0$ and $\widehat{\mathbf{B}}_r = \mathbf{G}_r^*$.
- Incorporating the interpolation conditions, we require:

$$\boxed{\mathbf{Q}_r > 0, \quad \mathbf{R}_r \geq 0, \quad \text{and} \quad \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + 2\mathbf{R}_r = \mathbf{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbf{G}_r}$$

# Special case: passive (pH) systems

Necessary and sufficient condition for a passive interpolatory system:

$$\boxed{\mathbf{Q}_r > 0, \quad \mathbf{R}_r \geq 0, \quad \text{and} \quad \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + 2\mathbf{R}_r = \mathbf{G}_r^*\mathbb{B}_r + \mathbb{B}_r^*\mathbf{G}_r}$$

Recall the Lyapunov operator: $\mathcal{L}_\Sigma(\mathbf{M}) = \bar{\Sigma}\mathbf{M} + \mathbf{M}\Sigma$

- $$\mathbf{Q}_r + 2\underbrace{\mathcal{L}_\Sigma^{-1}(\mathbf{R}_r)}_{\geq 0} = \underbrace{\mathcal{L}_\Sigma^{-1}(\mathbf{G}^*\mathbb{B}_r + \mathbb{B}_r^*\mathbf{G})}_{\mathbf{Q}_0}$$
- Notice that $\mathbf{Q}_0 = \mathcal{L}_\Sigma^{-1}(\mathbf{G}^*\mathbb{B}_r + \mathbb{B}_r^*\mathbf{G})$, is determined from data, so if $\mathbf{Q}_0$ fails to be positive definite then it will be impossible for $\mathbf{Q}_r$ to be positive definite and the original system could not have been passive.
- Conversely, if $\mathbf{Q}_0$ is positive definite, then there will be a convex family of positive-definite/semidefinite pairs $(\mathbf{Q}_r, \mathbf{R}_r)$ that satisfy $\mathbf{Q}_r + 2\mathcal{L}_\Sigma^{-1}(\mathbf{R}_r) = \mathbf{Q}_0$.

# Special case: passive (pH) systems

Consider the SISO case:
$\mathbf{G}^*\mathbb{B}_r = \mathbf{g}_r\mathbf{e}^T$ where $\mathbf{g}_r = [\mathcal{G}(\overline{\sigma_1}), \ldots \mathcal{G}(\overline{\sigma_r})]^T \in \mathbb{C}^r$ and $\mathbf{e}^T = [1,\ 1,\ \ldots,\ 1]$.

$$\boxed{\mathbf{Q}_r + 2\mathcal{L}_\Sigma^{-1}(\mathbf{R}_r) = \mathcal{L}_\Sigma^{-1}(\underbrace{\mathbf{g}_r\mathbf{e}^T + \mathbf{e}\mathbf{g}_r^*}_{\mathbf{F}}) = \mathbf{Q}_0 > 0}$$

- Since $\mathbf{Q}_0$ solves $\mathcal{L}_\Sigma(\mathbf{Q}_0) = \mathbf{F}$, with $\mathrm{rank}(\mathbf{F}) = 2$, $\mathbf{Q}_0$ will tend to have very rapidly decaying singular values. Thus, $\mathbf{Q}_r$, $\mathbf{R}_r$, and $\mathcal{L}_\Sigma^{-1}(\mathbf{R}_r)$ *will tend to have rapidly decaying singular values as well*.
- Approximate $\mathbf{R}_r$ with a low rank matrix, $\mathbf{R}_r = \sum_k \alpha_k \mathbf{M}_k$ with $\mathbf{M}_k \geq 0$ (rank one) and $\alpha_k \geq 0$ (variable, but potentially rapidly decaying).
- Parameterize $\mathbf{R}_r$ (and hence $\mathbf{Q}_r$) via $\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_r\}$

Model $\mathbf{R}_r$ as $\mathbf{R}_r = \sum_{\ell=1}^{p} \alpha_\ell \mathbf{M}_\ell$ where $p \leq r$ and $\mathbf{M}_\ell$ are spectral projectors associated with the *analytic center* of the LMI constraint: $\mathbf{R}_r = -\frac{1}{2}(\mathbf{Q}_r \mathbf{A}_r + \mathbf{A}_r^T \mathbf{Q}_r) \geq 0$,

$$\mathbf{R}_{ctr} = \text{argmax} \left\{ \text{logdet}(\mathbf{R}_r) \;\middle|\; \begin{array}{c} \mathbf{R}_r \geq 0 \\ \mathcal{L}_\Sigma^{-1}(\mathbf{R}_r) \leq \frac{1}{2}\mathbf{Q}_0 \end{array} \right\}$$

Then writing the spectral decomposition: $\mathbf{R}_{ctr} = \sum_{\ell=1}^{r} \lambda_\ell \mathbf{M}_\ell$, parameterize $\mathbf{R}_r$ as $\mathbf{R}_r = \sum_{\ell=1}^{r} \alpha_\ell \mathbf{M}_\ell$.

- Form a quadratic model of the $\mathcal{H}_2$-error wrt the $\mathbf{R}_r$-parameters $\alpha_1, \alpha_2, \ldots \alpha_r$
  $\|\mathcal{G} - \mathcal{G}_r\|_{\mathcal{H}_2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{G}(\imath\omega) - \mathcal{G}_r(\imath\omega)\|_F^2 \, d\omega \right)^{1/2}$
- Additional $\mathcal{G}$ evaluations can be avoided by aggregating earlier $\mathcal{G}$ evaluations into intermediate-order Löwner or VF models, $\mathcal{G}_*$.

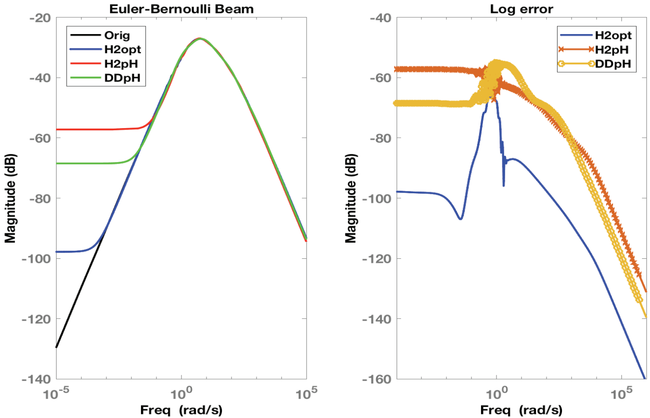
## Algorithm (Data-driven MOR for pH systems (DDPH))

A *Find (e.g., with RlznIndIRKA) a locally $\mathcal{H}_2$-optimal model $\mathcal{G}_*$ with interpolation points $\{\sigma_1, \sigma_2, \dots, \sigma_r\} \subset \mathbb{C}^+$,*

B

1. *Evaluate* $\mathbf{g}_r = [\mathcal{G}(\overline{\sigma_1}),\ \mathcal{G}(\overline{\sigma_2}),\ \dots,\ \mathcal{G}(\overline{\sigma_r})]^T$
2. *Solve* $\mathcal{L}_\Sigma(\mathbf{Q}_0) = \mathbf{g}_r\mathbf{e}^* + \mathbf{e}\mathbf{g}_r^*$ *for* $\mathbf{Q}_0 \in \mathbb{C}^{r\times r}$*.*
   *If* $\mathbf{Q}_0$ *is not positive-definite, then stop.* $\mathcal{G}$ *is not passive.*
3. *Find analytic center,* $\mathbf{R}_{ctr}$*, of LMI:* $\mathcal{L}_\Sigma^{-1}(\mathbf{R}) \leq \mathbf{Q}_0$ *with* $\mathbf{R} \geq 0$ *and resolve* $\mathbf{R}_{ctr}\mathbf{z}_k = \lambda_k \mathbf{z}_k$*, for* $k = 1, \dots r$*.*
4. *For* $\mathbf{M}_k = \mathcal{L}_\Sigma^{-1}(\mathbf{z}_k\mathbf{z}_k^*)$*, compute quadratic model of* $\|\mathcal{G}_* - \mathcal{G}_r\|_{\mathcal{H}_2}$ *with respect to* $\{\alpha_1, \alpha_2, \dots \alpha_r\}$ *for* $\mathcal{G}_r = \mathbf{g}_r^*((\mathbf{Q}_0 - 2\sum_k \alpha_k \mathbf{M}_k)(s - \Sigma) + \mathbf{g}_r\mathbf{e}^T)^{-1}\mathbf{g}_r$*.*
5. *Solve the SDP,*
   $$\min_{\alpha} \gamma^T\alpha + \frac{1}{2}\alpha^T\Gamma\alpha \quad \textit{subject to} \quad \mathbf{Q}_0 \geq 2\sum_k \alpha_k \mathbf{M}_k$$
6. *repeat...*

# ROM for Composite Beam: reduction order 8




"$H_2$opt" is (nonintrusive) $H_2$-optimal interpolatory MOR (does not preserve pH).
"$H_2$pH" is an (intrusive) interpolatory projection method that preserves pH structure.
"DDpH" is the present (nonintrusive) interpolatory MOR method that preserves pH.

For comparison: optimal $\mathcal{H}_2$ model produces relative $\mathcal{H}_2$ error 1.89e-3

# Extension to *structurally passive* nonlinear systems

Linear: $$\boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{z}} &= (\mathbf{J}-\mathbf{R})\mathbf{z} + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{z} \end{aligned}}$$ with $\mathbf{Q} > \mathbf{0}$, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.

# Extension to *structurally passive* nonlinear systems

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$$\boxed{\begin{aligned}\mathbf{Q}\dot{\mathbf{z}} &= (\mathbf{J}-\mathbf{R})\mathbf{z} + \mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{z}\end{aligned}}$$

with $\mathbf{Q} > \mathbf{0}$, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.

Nonlinear case:
$$\boxed{\begin{aligned}[\nabla^2\mathcal{E}(\mathbf{z})]\dot{\mathbf{z}} &= (\mathbf{J}(\mathbf{z}) - \mathbf{R}(\mathbf{z}))\mathbf{z} + \mathbf{C}(\mathbf{z})^T\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{z})\,\mathbf{z}\end{aligned}}$$

with $\mathcal{E}(\mathbf{z})$ uniformly convex, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.
$\mathbf{J}$, $\mathbf{R}$, and $\mathbf{C}$ could all depend on $\mathbf{z}$.

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with $\mathcal{E}(\mathbf{z})$ uniformly convex, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.
$\mathbf{J}$, $\mathbf{R}$, and $\mathbf{C}$ could all depend on $\mathbf{z}$.

Alternative (conjugate) representation: (Legendre transformation)
Define $\mathbf{x} = \nabla\mathcal{E}(\mathbf{z})$ and $H(\mathbf{x}) = \sup_{\mathbf{z}}\left(\mathbf{x}^T\mathbf{z} - \mathcal{E}(\mathbf{z})\right) \implies \mathbf{z} = \nabla H(\mathbf{x})$.

Then $$\boxed{\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}}H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}}H(\mathbf{x}) \end{aligned}}$$ $H(\mathbf{x})$ is a *storage* function.

$H(\mathbf{x})$ is uniformly convex, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.
$\mathbf{J}$, $\mathbf{R}$, and $\mathbf{C}$ now all depend (potentially) on $\mathbf{x}$.

General "port-Hamiltonian" representation of the system.

# Alternative supply rates: contractive systems

Pick $\gamma > 0$ and take the supply rate to be:

$$w(\mathbf{y}(t), \mathbf{u}(t)) = \frac{1}{2}\left(\gamma^2\|\mathbf{u}(t)\|^2 - \|\mathbf{y}(t)\|^2\right) = \frac{1}{2}\left(\ \mathbf{y}^T\ \mathbf{u}^T\ \right)\begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$$

(defining $\mathbf{M} = \gamma^2\mathbf{I}$, $\mathbf{N} = -\mathbf{I}$ and $\Omega = \mathbf{0}$)

and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \ \text{ for } \ \mathbf{Q} > 0.$$

- The system is $\gamma$-**contractive** with the storage function $\mathsf{H}(\mathbf{x})$, if and only if $\mathbf{Q}$ is a positive-definite solution to the LMI:

$$\begin{bmatrix} \mathbf{Q}\mathbf{A} + \mathbf{A}^T\mathbf{Q} + \mathbf{C}^T\mathbf{C} & \mathbf{Q}\mathbf{B} \\ \mathbf{B}^T\mathbf{Q} & -\gamma^2\mathbf{I} \end{bmatrix} \leq 0 \Leftrightarrow \quad \mathbf{Q}\mathbf{A} + \mathbf{A}^T\mathbf{Q} + \mathbf{C}^T\mathbf{C} + \frac{1}{\gamma^2}\mathbf{Q}\mathbf{B}\mathbf{B}^T\mathbf{Q} \leq 0$$

(Riccati Matrix Inequality)

- If $\mathcal{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ is the transfer function for the system then the system is $\gamma$-contractive if and only if $\|\mathcal{G}\|_{\mathcal{H}_\infty} \leq \gamma$. This is an important property to insure when designing model-based stabilizing controllers that are robust to model uncertainty.

# Special case: Nevanlinna-Pick Interpolation

## Nevanlinna-Pick Interpolation problem

Given a set of distinct points, $\{\sigma_1, ..., \sigma_r\}$ in the open right half-plane and target values, $\{\gamma_1, ..., \gamma_r\} \subset \mathbb{C}$ with $\max_i |\gamma_i| \leq 1$.
Find a function, $F(s)$, analytic in the right half plane such that

(a) $F(\sigma_i) = \gamma_i$ for $i = 1, ..., r$, and (b) $\max_{\omega \in \mathbb{R}} |F(\boldsymbol{i}\omega)| \leq 1$.

Such an $F(s)$ exists if and only if the $r \times r$ "Pick matrix": $\mathbf{P}$,

with components $\mathbf{P}_{ij} = \dfrac{1 - \gamma_i \bar{\gamma}_j}{\sigma_i + \bar{\sigma}_j}$ is positive definite.

- Property (a) is a SISO interpolation condition on $F$ (viewed as a transfer function).
- Property (b) asserts that $F$ is contractive, i.e. dissipative wrt the supply rate:

$$w(y(t), u(t)) = \tfrac{1}{2}\left(|u(t)|^2 - |y(t)|^2\right) = \tfrac{1}{2}\begin{pmatrix} y & u \end{pmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$

# Special case: Nevanlinna-Pick Interpolation

Define $\Sigma = \text{diag}(\sigma_1, ..., \sigma_r)$, $\mathbb{B}_r = [1, \ldots, 1] = \mathbf{e}^T$, and $\mathbf{G}_r = [\gamma_1, ..., \gamma_r] \stackrel{\text{def}}{=} \mathbf{g}_r^T$ and denote $\widehat{\mathbf{B}}_r = \hat{\mathbf{b}}_r \in \mathbb{C}^r$.

The solution requires: $\mathbf{Q}_r > 0$ and $\begin{cases} \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + 2\mathbf{R}_r = \hat{\mathbf{b}}_r\mathbf{e}^T + \mathbf{e}\hat{\mathbf{b}}_r^* \\ \begin{bmatrix} -2\mathbf{R}_r + \mathbf{g}_r\mathbf{g}_r^* & \hat{\mathbf{b}}_r \\ \hat{\mathbf{b}}_r^* & -1 \end{bmatrix} \leq 0 \end{cases}$

Equivalently,

- Dissipitivity LMI reduces to: $\mathbf{g}_r\mathbf{g}_r^* + \hat{\mathbf{b}}_r\hat{\mathbf{b}}_r^* \leq 2\mathbf{R}_r$.
- Incorporating the interpolation conditions, we require:

$$\mathbf{Q}_r > 0 \quad \text{and} \quad \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma + \mathbf{g}_r\mathbf{g}_r^* + \hat{\mathbf{b}}_r\hat{\mathbf{b}}_r^* \leq \hat{\mathbf{b}}_r\mathbf{e}^T + \mathbf{e}\hat{\mathbf{b}}_r^*$$

or equivalently,

$$\boxed{\mathbf{Q}_r > 0 \quad \text{and} \quad \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma \leq (\mathbf{e}\mathbf{e}^T - \mathbf{g}_r\mathbf{g}_r^*) - (\hat{\mathbf{b}}_r + \mathbf{e})(\hat{\mathbf{b}}_r + \mathbf{e})^*}$$

# Special case: Nevanlinna-Pick Interpolation

Necessary and sufficient condition for a contractive interpolatory system:

$$\boxed{\mathbf{Q}_r > 0 \quad \text{and} \quad \bar{\Sigma}\mathbf{Q}_r + \mathbf{Q}_r\Sigma \leq (\mathbf{e}\mathbf{e}^T - \mathbf{g}_r\mathbf{g}_r^*) - (\hat{\mathbf{b}}_r + \mathbf{e})(\hat{\mathbf{b}}_r + \mathbf{e})^*}$$

Define the Lyapunov operator: $\boxed{\mathcal{L}_\Sigma(\mathbf{M}) = \bar{\Sigma}\mathbf{M} + \mathbf{M}\Sigma}$

- $\mathcal{L}_\Sigma : \mathbb{C}^{r\times r}_{Herm} \to \mathbb{C}^{r\times r}_{Herm}$ bijectively, and the cone of positive/negative semidefinite matrices is preserved by $\mathcal{L}_\Sigma^{-1}$.
- Thus, we have that

$$\mathbf{Q}_r \leq \underbrace{\mathcal{L}_\Sigma^{-1}(\mathbf{e}\mathbf{e}^T - \mathbf{g}_r\mathbf{g}_r^*)}_{\text{Pick matrix !!}} - \underbrace{\mathcal{L}_\Sigma^{-1}((\hat{\mathbf{b}}_r + \mathbf{e})(\hat{\mathbf{b}}_r + \mathbf{e})^*}_{\text{positive semidefinite}}$$

- If the Pick matrix is *not* positive definite then it is impossible for $\mathbf{Q}_r$ to be positive definite; no contractive interpolant can exist.
- Conversely, if the Pick matrix is positive definite, there will be an infinite number of solutions parameterized by $\mathbf{Q}_r$, $\mathbf{R}_r$, and $\hat{\mathbf{b}}_r$.

# Extending spectral zero interpolation

Antoulas⋆ developed an interpolatory projection approach for passivity-preserving MOR, that ...

- Does not require explicit solution of Luré LMI; but...
- Requires extraction of $r$-dim anti-stable deflating subspace:

$$\boxed{\begin{bmatrix} \mathbf{0} & \mathbf{A} & \mathbf{B} \\ \mathbf{A}^T & \mathbf{0} & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_r \\ \mathbf{X}_r \\ \mathbf{Z}_r \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_r \\ \mathbf{X}_r \\ \mathbf{Z}_r \end{bmatrix} \mathbf{M}_r} \quad \text{with } \sigma(\mathbf{M}_r) \subset \mathbb{C}_+.$$

- When $r = n$ and $(\mathbf{A}, \mathbf{B})$ is stabilizable, then $\mathbf{X}_n \in \mathbb{R}^{n \times n}$ is invertible and $\mathbf{Q} = \mathbf{Y}_n \mathbf{X}_n^{-1}$ is a storage function (implicitly defined) associated with the supply rate: $\mathbf{y}^T \mathbf{u}$. (the system is passive).
- Reduced (passive) model is defined by $\mathcal{G}_r(s) = \mathbf{C}_r(s\mathbf{Q}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$,
  $\mathbf{Q}_r = \mathbf{X}_r^T \mathbf{Q} \mathbf{X}_r = \mathbf{Y}_r^T \mathbf{X}_r, \quad \mathbf{A}_r = \mathbf{Y}_r^T \mathbf{A} \mathbf{X}_r, \quad \mathbf{B}_r = \mathbf{Y}_r^T \mathbf{B}, \quad \mathbf{C}_r = \mathbf{C} \mathbf{X}_r.$
- $\mathcal{G}_r$ interpolates $\mathcal{G}$ at $2r$ points (spectral zeros) to produce an order $r$ passive model: $\mathcal{G}(\pm\lambda) = \mathcal{G}_r(\pm\lambda)$ for $\lambda \in \sigma(\mathbf{M}_r)$ .

⋆ Antoulas, A. C. (2005). A new result on passivity preserving model reduction. *Systems & Control Letters,* 54(4), pp 361-374.

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# Extending spectral zero interpolation

- For a general quadratic supply rate:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \frac{1}{2}\begin{pmatrix} \mathbf{y}(t) & \mathbf{u}(t) \end{pmatrix} \begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix} \begin{pmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{pmatrix}$$

with $\mathbf{M} \geq 0 \quad \mathbf{N} \geq 0$ can consider instead

$$\boxed{\begin{bmatrix} \mathbf{0} & \mathbf{A} & \mathbf{B} \\ \mathbf{A}^T & -\mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{C}^T\Omega \\ \mathbf{B}^T & \Omega^T\mathbf{C} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_r \\ \mathbf{X}_r \\ \mathbf{Z}_r \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_r \\ \mathbf{X}_r \\ \mathbf{Z}_r \end{bmatrix} \mathbf{M}_r}$$

- Same advantages/disadvantages as for passivity-preserving spectral zero interpolation.

- Seek a formulation that is data-driven...

  But see partial progress in this direction by Benner, Goyal, and van Dooren($\star$). They formulate passive reduced models by approximating spectral zeros of the full system from an intermediate Loewner realization.

$\star$ Peter Benner, Pawan Goyal, and Paul van Dooren, "Identification of Port-Hamiltonian Systems from Frequency Response Data" (2019) arXiv:1911.00080

# Summary

- Reviewed basic notions of dissipative systems for LTI systems.
  - Key point: dissipativity is an exogenous property tied to a specified supply rate, not tied to a particular realization.
  - A particular realization gives rise to a family of storage functions (parameterized by solutions to an LMI).
- Introduced an (intrusive) interpolatory projection method that preserves dissipative structure.
  - $+$ Pro: Allows arbitrary state-space projection - gives potential for high-fidelity
  - $-$ Con: Requires knowledge of a storage function (intractable for large order)
  - $?$ Extension of "spectral zero" approach for passivity preservation to arbitrary (quadratic) supply rates. Is a data-driven formulation feasible ?
- Introduced a "data-driven" model reduction strategy that preserves stability and passivity.
  - $+$ Only transfer function evaluations are needed ("data-driven")
  - $+$ Formulation leads to convex programming problems of small size.